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GAUSSIAN APPROXIMATIONS TO SERVICE PROBLEMS:

A COMMUNICATION SYSTEM EXAMPLE

by

D. P. Gaver

and

J. P. Lehoczky

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Messages arrive at a group of service channels in accordance with a time-dependent Poisson process. An arrival either (i) immediately begins k-stage Markovian service if an empty channel is reached, or (ii) balks and enters a retrial population if the channel sought is busy. Diffusion approximations to the number of messages in service (each stage) and in the retrial population are derived by writing stochastic differential

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(I+0) equations. Steady-state distributions are found and compared with certain simulation results.

GAUSSIAN APPROXIMATIONS TO SERVICE PROBLEMS: A COMMUNICATION SYSTEM EXAMPLE

D. P. Gaver*
Naval Postgraduate School
Monterey, California

J. P. Lehoczky**
Carnegie-Mellon University
Pittsburgh, Pennsylvania

1. Introduction.

Any reasonable model of a complex communication or other service system requires consideration of several interacting populations. Consequently, it becomes necessary to name the state of each population, and to describe the state of the entire system in terms of a vector-valued state variable. With rare exceptions very little information can then be derived in a direct manner, e.g. by postulating that the modelling process is multidimensional birth-death or, more generally, Markov, and deriving mathematical expressions for steady-state probabilities, etc., as exemplified by Feller [3] I, Chap XVII, and in many papers. Exceptions do occur but seem rare, see the cyclic queue model of Gordon and Newell [5], and related work by Whittle [10] and by Kingman [6]. Consequently, it is tempting to devise approximate diffusion models for such processes; previous work with this intent has been done by Schach and McNeill [8], McNeill [7],

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and the approximation studied by Barbour [2]. Our paper, [4], indicates how this may be done for a transitory situation. In this paper we study a communication system that consists of c channels, sought for by arriving messages that balk if they encounter an occupied channel. Balking messages enter a distinct (retrial) population, or populations, from which they may eventually either defect, or again apply for service on one of the channels. Previous work in this problem area along classical point-process model lines is described by Riordan [9]. Our purpose here is to indicate how stochastic differential equations may be used to write down a model, and how useful information may then be derived. In order to illustrate the degree of accuracy achieved we have conducted some sample simulations, the results of which generally support the approximation method.

2. The Model.

The model structure is depicted in Figure 1. Input to a service center arrives according to a Poisson process with intensity $c\lambda(t)$. The service center consists of k distinct compartments and a total of c channels or servers. The service process on the $i^{\underline{th}}$ compartment is Markovian $(\mu_i(t))$; i.e. if a "customer," here message, is undergoing stage-i service at t it terminates within $(t,t+dt)$ with probability $\mu_i(t)dt + o(dt)$, independently of previous process history. Upon completion of the $i^{\underline{th}}$ service function a message proceeds immediately to the $(i+1)^{\underline{st}}$. The channels are considered separate, and if any compartment of a channel is occupied then no other message can enter that channel. Consequently, the service center has a capacity of c messages at any one time, and a single message has a total service time which is the sum of k independent but time-dependent exponentials. If $\mu_i(t) = \mu$, $i = 1,\ldots,k$ then the message service time is Gamma (k,μ). In fact, the compartments are introduced in order to permit the modelling of non-exponential service.

We postulate that a message selects a channel uniformly at random from among the c possible channels. If that channel is busy, the message is denied immediate access to service. Otherwise, the message occupies the selected channel until service in all k compartments has been completed. The random selection of channels and temporary denial of service goes unrecognized in the context of classical queueing problems; however, such assumptions are quite appropriate for certain communication situations (telephone

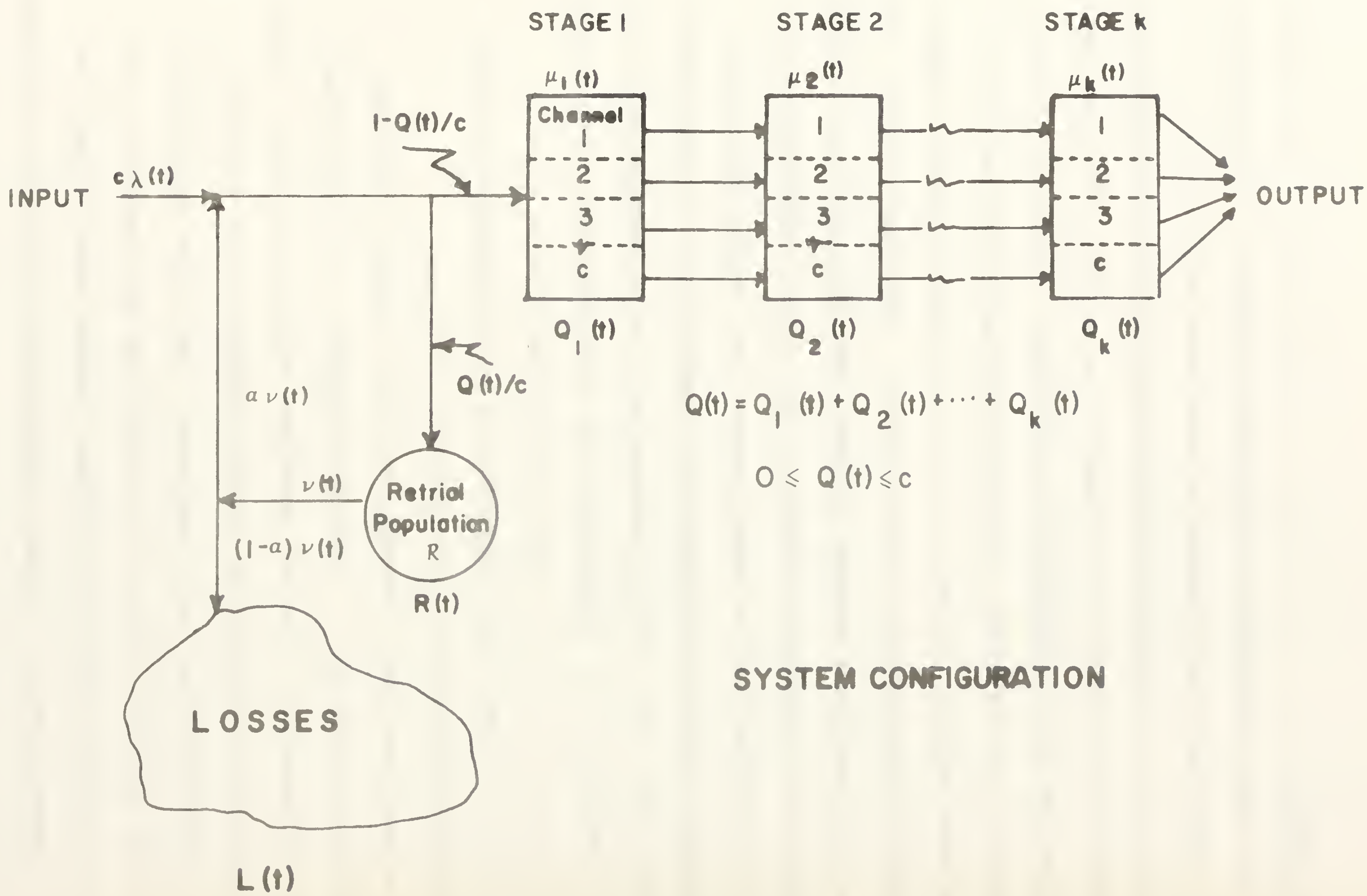
STAGE 1
STAGE 2
STAGE k
$\mu_1(t)$
$\mu_2(t)$
$\mu_k(t)$
Channel
1
2
3
c
INPUT
$c\lambda(t)$
$1-Q(t)/c$
OUTPUT
$Q_1(t)$
$Q_2(t)$
$Q_k(t)$
$Q(t)/c$
$\alpha\nu(t)$
$\nu(t)$
$(1-\alpha)\nu(t)$
Retrial Population R
R(t)
LOSSES
L(t)
$Q(t) = Q_1(t) + Q_2(t) + \cdots + Q_k(t)$
$0 \leq Q(t) \leq c$
SYSTEM CONFIGURATION

Figure 1

or air-traffic control, for example). In such cases the customer may not automatically select an empty channel, but must use whatever channel is appropriate, whether busy or not. If channels tend to be used equally, the random selection is then appropriate.

If a customer is not serviced there is no physical queue. A telephone caller receiving a busy signal can only hang up, perhaps with the intent to call again shortly. An airline pilot or submarine attempting to contact a busy traffic controller or shore communication facility must try at a later time, possibly to be denied service again. We assume that messages or customers who are denied service enter a category R. Such customers may retry or recall at a later time. These customers reapply for service after being in state R for a Markovian $(\nu(t))^{-1}$ period of time. In most cases $\nu(t)$ would be rather large if the customer must receive service (air-traffic control), but might be small in the case of telephone calls.

In many situations customer discouragement will occur, expecially if repeated requests for service are denied. We adopt a simple model of customer discouragement by assuming a binomial probability $1-\alpha$ that a customer upon leaving R decides to leave the system and is lost. It is clear that a vast array of models can be introduced to represent customer discouragement. The most general sort might include states $\{R_i\}_{i=1}^{\infty}$ with R_i signifying that the customer has been denied service i times. The holding time in R_i is Markovian $(\nu_i(t))^{-1}$, and $1-\alpha_i$ is the probability of loss. We discuss only the simple model mentioned earlier with one state R, but the reader may notice that the analysis used can easily be extended to handle the most general case with states $\{R_i\}_{i=1}^{\infty}$.

We wish to describe the behavior of this system by calculating system characteristics such as the fraction of customers lost, the utilization of the c channels, and the number of customers in state R, all as a function of the system parameters. Standard Markov chain methods can be used on this model owing to the Markovian assumptions. Unfortunately, the state of the system is a $k+2$ dimensional vector. While a Markov chain analysis is straightforward, it must be carried out numerically. This makes it difficult to assess the influence of system parameters on the quantities of interest. Furthermore, it is difficult to deduce information about the transient behavior of the system and to handle non-stationary transition probabilities using Markov chain techniques. For these reasons we adopt the method of diffusion approximations as an approach. This consists of letting $c \to +\infty$ and treating the resulting random processes as the sum of a deterministic process plus an additive noise (diffusion) process. This technique allows one to compute the quantities of interest as a function of system parameters and to describe the transient behavior of the system. If we assume the transitions are stationary, then techniques from the theory of stationary processes, especially spectral analysis, can also be applied. While all results are exact only in the limit as $c \to \infty$, we present comparisons between simulated systems with $c = 10$ and 20, and diffusion approximations. The comparisons will show that diffusion approximations may offer surprising accuracy even for c as small as 10. Thus the method described is quite appealing as an approximation tool in the present problems, and in many others as well.

3. Diffusion Approximation of the System.

We introduce the following notation:

$Q_i(t)$ = number of messages or customers at service stage i, at time t, $i = 1,\ldots,k$,

$Q(t) = \sum_{i=1}^{k} Q_i(t)$ = number of customers receiving service at time t,

$R(t)$ = number of customers in state R at time t,

$L(t)$ = cumulative number of customers lost by time t.

A customer arriving at the service center at time t selects a channel at random, hence with probability $Q(t)/c$ is denied immediate service, and with probability $1 - Q(t)/c$ begins service. It is straightforward to describe this $k+2$ dimensional system and its transition probabilities over the interval $(t,t+dt)$; however, we choose to do this approximately to terms of order dt using techniques from the theory of stochastic differential equations; see Arnold [1], and also Gaver, Lehoczky and Perlas [4]. Using the notation $dX(t) = X(t+dt) - X(t)$ for a stochastic process $\{X(t), t \geq 0\}$ we express the evolution of our process as follows:

$$dQ_1(t) = (\lambda(t)c+\alpha\nu(t)R(t))(1 - Q(t)/c)dt - \mu_1(t)Q_1(t)dt$$

$$+ \sqrt{\lambda(t)c(1 - Q(t)/c)}\; dW_\lambda(t) + \sqrt{\alpha\nu(t)R(t)(1 - Q(t)/c)}\; dW_R(t)$$

$$- \sqrt{\mu_1(t)Q_1(t)}\; dW_{Q_1}(t) \qquad (3.1)$$

$$dQ_i(t) = \mu_{i-1}(t)Q_{i-1}(t)dt - \mu_i(t)Q_i(t)dt + \sqrt{\mu_{i-1}(t)Q_{i-1}(t)}\, dW_{Q_{i-1}}(t)$$

$$- \sqrt{\mu_i(t)Q_i(t)}\, dW_{Q_i}(t) \quad \text{for} \quad i = 2,\ldots,k$$

$$dR(t) = -(1-\alpha+\alpha(1-Q(t)/c))\nu(t)R(t)dt + \lambda(t)c(Q(t)/c)dt$$

$$- \sqrt{(1-\alpha+\alpha(1-Q(t)/c))\nu(t)R(t)}\, dW_R(t) + \sqrt{\lambda(t)c(Q(t)/c)}\, dW_\lambda(t)$$

$$dL(t) = (1-\alpha)\nu(t)R(t)dt + \sqrt{(1-\alpha)\nu(t)R(t)}\, dW_R(t).$$

In equations (3.1) the terms $dW_{Q_i}(t)$, $dW_R(t)$, and $dW_\lambda(t)$ are the "derivatives" of independent standard Wiener processes, and as such are usually called Gaussian white noise. Other approximations are possible, but are not pursued here. We mention, for example, using Poisson white noise where the standard Wiener process is replaced by a Poisson process with zero drift.

A word about the derivation of our equations (3.1) is in order. Consider that for the occupancy of the first service compartment, $Q_1(t)$: conditional on the values of $Q_1(t)$ and $R(t)$ the <u>drift</u> or mean change of Q_1 in time dt is (a) positive by the amount $(\lambda(t)c+\alpha\nu(t)R(t))(1-\frac{Q(t)}{c})dt$, where the first term represents the expected number of arrivals in $(t,t+dt)$, and the other represents the probability of acceptance into service in compartment $i=1$, while (b) the second, negative, term $-\mu_1(t)Q_1(t)dt$, represents the expected number departing Q_1 in $(t,t+dt)$; hence the difference is the net expected increase in Q_1. Now for dt small we represent the fluctuating (diffusion) component of input by (c): $\sqrt{\lambda(t)c(1-Q(t)/c)}\, dW_\lambda(t)$, where the scale factor is the standard deviation of a Poisson process with mean $\lambda(t)c(1-Q(t)/c)$ and $dW_\lambda(t)$ is $\mathcal{N}(0,dt)$, plus (d) a corresponding but independent term representing arrivals from $\mathcal{R}$, minus (e) another corresponding term representing departure from Q_1. Although this derivation is heuristic, the rationale is simply that in an interval of length dt the various conditionally Poisson components of change are approximately normal, owing to the fact that c is presumed large.

We wish next to view the state of the system $(Q_1(t),\ldots,Q_k(t),R(t),L(t))$ as the sum of a deterministic process plus a noise or diffusion process. To accomplish this we introduce the standardized noise variables:

$$
\begin{aligned}
X_i(t) &= \frac{Q_i(t) - cq_i(t)}{\sqrt{c}} \qquad i = 1,\ldots,k; \quad X(t) = \sum_{i=1}^{k} X_i(t) \\
Y(t) &= \frac{R(t) - cr(t)}{\sqrt{c}} \\
Z(t) &= \frac{L(t) - c\ell(t)}{\sqrt{c}} \\
q(t) &= \sum_{i=1}^{k} q_i(t)
\end{aligned}
\tag{3.2}
$$

We then write $(Q_1(t),\ldots,Q_k(t),R(t),L(t))$ in vector fashion as

$$
c(q_1(t),q_2(t),\ldots,q_k(t),r(t),\ell(t)) + c(X_1(t),X_2(t),\ldots,X_k(t),Y(t),Z(t)),
$$

the first term is the deterministic approximation, the last is the noise process. We substitute definitions (3.2) into (3.1) and divide the resulting equations by $\sqrt{c}$. The results are, to terms of order 1 in c, expressible as

$$dX_1(t) = \{(1-q(t))\alpha\nu(t)Y(t) - (\lambda(t)+\alpha\nu(t)r(t))X(t) - \mu_1(t)X_1(t)\}dt$$

$$+ \sqrt{\lambda(t)(1-q(t))}\, dW_\lambda + \sqrt{\alpha\nu(t)r(t)(1-q(t))}\, dW_R - \sqrt{\mu_1(t)q_1(t)}\, dW_{Q_1}$$

$$- \sqrt{c}\{q_1'(t) - (\lambda(t)+\alpha\nu(t)r(t))(1-q(t)) + \mu_1(t)q_1(t)\}dt,$$

$$dX_i(t) = \{\mu_{i-1}(t)X_{i-1}(t) - \mu_i(t)X_i(t)\}dt + \sqrt{\mu_{i-1}(t)q_{i-1}(t)}\, dW_{Q_{i-1}}$$

$$- \sqrt{\mu_i(t)q_i(t)}\, dW_{Q_i} - \sqrt{c}\{q_i'(t) - \mu_{i-1}(t)q_{i-1}(t) + \mu_i(t)q_i(t)\}dt$$

$$\text{for } i = 2,\ldots,k, \qquad (3.3)$$

$$dY(t) = \{-[(1-\alpha q(t))\nu(t)]Y(t) + [\lambda(t) + \alpha\nu(t)r(t)]X(t)\}dt$$

$$- \sqrt{(1-\alpha q(t))\nu(t)r(t)}\, dW_R + \sqrt{\lambda(t)q(t)}\, dW_\lambda$$

$$- \sqrt{c}\{r'(t) + (1-\alpha q(t))\nu(t)r(t) - \lambda(t)q(t)\}dt,$$

and

$$dZ(t) = (1-\alpha)\nu(t)Y(t)dt + \sqrt{(1-\alpha)\nu(t)r(t)}\, dW_R$$

$$- \sqrt{c}\{\ell'(t) - (1-\alpha)\nu(t)r(t)\}dt.$$

We now let $c \to \infty$ in equations (3.3). Clearly, in all cases the $\sqrt{c}$ term must be identically 0, or else the equations explode. Setting the $\sqrt{c}$ term to 0 we derive a system of ordinary differential equations satisfied by the deterministic approximation:

$$q_1'(t) = (\lambda(t)+\alpha\nu(t)r(t))(1-q(t)) - \mu_1(t)q_1(t)$$

$$q_i'(t) = \mu_{i-1}(t)q_{i-1}(t) - \mu_i(t)q_i(t) \qquad i = 2,\ldots,k \qquad (3.4)$$

$$r'(t) = -(1-\alpha q(t))\nu(t)r(t) + \lambda(t)q(t)$$

$$\ell'(t) = (1-\alpha)\nu(t)r(t)$$

It is easy to find $q_i(t)$ in terms of $q_{i-1}(t)$ for $i = 2,\ldots,k$ by solving the second equation in (3.4).

$$q_i(t) = \int_0^t e^{-\int_s^t \mu_i(x)dx} \mu_{i-1}(s)q_{i-1}(s)ds + q_i(0)e^{-\int_0^t \mu_i(s)ds}$$

but no further explicit results seem attainable without simplification and further parameter specification. Of course numerical solution of the differential equations by computer is always possible, and may well prove to be the fastest route to useful information.

Steady State Behavior of the System

Suppose however that we let $t \to \infty$ and attempt to find a steady state solution. As $t \to \infty$ let $\lambda(t)$, $\nu(t)$, $\mu_i(t)$, $q_i(t)$, and $r(t)$ converge to λ, ν, μ_i, q_i, and r respectively. Then $q_i'(t)$ and $r'(t)$ all converge to 0 and the steady state equations become

$$
\begin{aligned}
0 &= (\lambda+\alpha\nu r)(1-q) - \mu_1 q_1 \\
0 &= \mu_{i-1} q_{i-1} - \mu_i q_i \qquad i = 2,\ldots,k \\
0 &= -(1-\alpha q)\nu r + \lambda q \\
\ell'(\infty) &= (1-\alpha)\nu r
\end{aligned}
\tag{3.5}
$$

Letting $\mu = 1/\sum_{i=1}^{k} \frac{1}{\mu_i}$ we find the steady-state values

$$
r(\infty) = \frac{\lambda q}{\nu(1-\alpha q)}, \qquad q(\infty) = \frac{\lambda+\mu-\sqrt{(\lambda+\mu)^2-4\lambda\alpha\mu}}{2\alpha\mu}
$$

The negative square root is used for q since $0 \le q \le 1$. Furthermore, the asymptotic loss rate, $\ell'(\infty)$, is given by $\frac{\lambda q(1-\alpha)}{1-\alpha q}$. The input rate is λ, so the output rate of those actually served is $\lambda\left(\frac{1-q}{1-\alpha q}\right)$, and the asymptotic loss fraction (the fraction of customers leaving without receiving service) is given by $(1-\alpha)q/(1-\alpha q)$.

We intend to carry out the subsequent analysis under the assumption that steady state conditions prevail. In this model the deterministic equations are nonlinear, hence there is a question of the stability of the system (in the sense of Liapunov). We must be concerned with the effect of a small perturbation when the system is in steady state. If the system is not stable then it will diverge from, rather than return to, steady state. The stability can be established when $\lambda(t)$, $\mu_i(t)$ and $\nu(t)$ converge to λ, μ_i, and ν by first linearizing equations (3.4). We omit $\ell(t)$ from the system since it clearly grows without limit and has no steady state value. We assume now that the parameters λ, μ_i, and ν are all constants, and express (3.4) in matrix form:

$$\begin{pmatrix} q_1'(t) \\ q_2'(t) \\ \cdot \\ \cdot \\ \cdot \\ q_k'(t) \\ r'(t) \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{pmatrix} + \begin{pmatrix} -(\lambda+\mu_1) & -\lambda & & -\lambda & \alpha\nu \\ \mu_1 & -\mu_2 & & 0 & 0 \\ 0 & \mu_2 & \cdots & \cdot & \cdot \\ \cdot & 0 & & \cdot & \cdot \\ \vdots & \vdots & & 0 & \cdot \\ 0 & 0 & & -\mu_k & 0 \\ \lambda & \lambda & & \lambda & -\nu \end{pmatrix} \begin{pmatrix} q_1(t) \\ \\ \cdot \\ \cdot \\ \cdot \\ q_k(t) \\ r(t) \end{pmatrix} \tag{3.7}$$

or

$$\underset{\sim}{X}'(t) = \underset{\sim}{c} + \underset{\sim}{A}\underset{\sim}{X}(t)$$

The system will be stable in the sense of Liapunov if the $k+1$ eigenvalues of the $\underset{\sim}{A}$ matrix have strictly negative real parts. The eigenvalues all have strictly negative real parts provided $\mu_i > 0$ and $\nu > 0$ for $i = 1,\ldots,k$. We prove this in the appendix.

We now turn to a description of the noise process. We assume deterministic equations (3.7) are satisfied. The representation (3.3) becomes in matrix form

$$d\underset{\sim}{U}(t) = \underset{\sim}{A}_t\underset{\sim}{U}(t)dt + \underset{\sim}{B}_t d\underset{\sim}{W}_t \tag{3.8}$$

where $\underset{\sim}{U}(t) = (X_1(t),\ldots,X_k(t),Y(t),Z(t))'$

$\underset{\sim}{W}(t) = (W_\lambda(t),W_R(t),W_{Q_1}(t),\ldots,W_{Q_k}(t))'$

$$\underset{\sim}{A}_t = \begin{pmatrix} -(\mu_1+\lambda+\alpha\nu r) & -(\lambda+\alpha\nu r) & \cdots & -(\lambda+\alpha\nu r) & \alpha\nu(rq) & 0 \\ \mu_1 & -\mu_2 & & 0 & 0 & 0 \\ 0 & \mu_2 & & \vdots & \vdots & \vdots \\ \vdots & 0 & & & & \\ \vdots & \vdots & & 0 & & \vdots \\ 0 & 0 & & -\mu_k & 0 & \vdots \\ \lambda+\alpha\nu r & \lambda+\alpha\nu r & & \lambda+\alpha\nu r & -(1-\alpha q)\nu & \\ 0 & 0 & & 0 & (1-\alpha)\nu & 0 \end{pmatrix}$$

and

$$\underset{\sim}{B}_t = \begin{pmatrix} \sqrt{\lambda(1-q)} & \sqrt{\alpha\nu r(1-q)} & -\sqrt{\mu q} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\mu q} & -\sqrt{\mu q} & & \vdots \\ 0 & \vdots & 0 & \sqrt{\mu q} & & \vdots \\ \vdots & \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & \vdots & \vdots & & -\sqrt{\mu q} \\ \sqrt{\lambda q} & -\sqrt{(1-\alpha q)\nu r} & & \vdots & & 0 \\ 0 & \sqrt{(1-\alpha q)\nu r} & 0 & 0 & & 0 \end{pmatrix}$$

In the above definitions of $\underset{\sim}{A}_t$ and $\underset{\sim}{B}_t$ we have omitted the argument t in $\lambda(t)$, $\mu_i(t)$, $\nu(t)$, $q(t)$, and $r(t)$ for notational convenience. As $t \to \infty$ $\underset{\sim}{A}_t$ and $\underset{\sim}{B}_t$ converge to $\underset{\sim}{A}$ and $\underset{\sim}{B}$ again given by (3.8); however, asymptotically $Z(t)$ is not of interest because $L(t) \to +\infty$. Furthermore, $\underset{\sim}{A}_t$ is singular. For these reasons we eliminate $Z(t)$ and reduce the dimension to $k+1$. The resulting stochastic differential equation becomes

$$d\underset{\sim}{V}(t) = \underset{\sim}{C}_t \underset{\sim}{V}(t)\,dt + \underset{\sim}{D}_t \, d\underset{\sim}{W}_t \qquad (3.9)$$

with $\underset{\sim}{V}(t) = (X_1(t),\ldots,X_k(t),Y(t))'$ and $\underset{\sim}{C}_t$ given by $\underset{\sim}{A}_t$ with the last row and column removed. $\underset{\sim}{D}_t$ is given by $\underset{\sim}{B}_t$ with the last row removed. Again $\underset{\sim}{C}_t$ and $\underset{\sim}{D}_t$ converge to $\underset{\sim}{C}$ and $\underset{\sim}{D}$ as $t \to \infty$.

The stochastic process $\underset{\sim}{V}(t)$ satisfying (3.9) is a non-stationary multivariate Ornstein-Uhlenbeck process. This process has been extensively studied, with many results recorded by Arnold ([1], Section 8.2). We shall make use of these results in what follows.

We may integrate (3.9) to find

$$\underset{\sim}{V}(t) = \underset{\sim}{V}(0) + \int_0^t \underset{\sim}{C}_t \underset{\sim}{V}(t)\,dt + \int_0^t \underset{\sim}{D}_t \, d\underset{\sim}{W}_t . \qquad (3.10)$$

The process $\underset{\sim}{V}(t)$ is Gaussian if $\underset{\sim}{V}(0)$ is either constant or itself Gaussian. This is clear from direct examination of (3.10). Suppose $\underset{\sim}{V}(t)$ is either constant or Gaussian. $\underset{\sim}{V}(t+dt)$ is the sum of $\underset{\sim}{V}(t) + \underset{\sim}{C}_t\underset{\sim}{V}(t)dt$, which is Gaussian, and another Gaussian variable. Thus $\underset{\sim}{V}(t+dt)$ will also be Gaussian and it remains to characterize the marginal moments of $\underset{\sim}{V}(t)$ as well as the covariance structure.

Let $\underset{\sim}{\mu}_t = E(\underset{\sim}{V}(t))$, $\underset{\sim}{\Sigma}_t = E((\underset{\sim}{V}(t)-\underset{\sim}{\mu}_t)(\underset{\sim}{V}(t)-\underset{\sim}{\mu}_t))'$ be the marginal mean and covariance of $\underset{\sim}{V}(t)$. It is shown in Arnold [1] that $\underset{\sim}{\mu}_t$ and $\underset{\sim}{\Sigma}_t$ satisfy first-order differential equations. First, the mean vector is described by

$$\dot{\underset{\sim}{\mu}}_t = \underset{\sim}{C}_t \underset{\sim}{\mu}_t \quad \text{with} \quad \underset{\sim}{\mu}_t = \underset{\sim}{V}(0) \qquad (3.11)$$

and, second, the covariance matrix Σ_t is the unique symmetric nonnegative definite solution of the matrix differential equation

$$\dot{\Sigma}_t = C_t \Sigma_t + \Sigma_t C_t' + D_t D_t' \quad \text{with} \quad \Sigma_0 = E((V(0)-\mu_0)(V(0)-\mu_0)'). \tag{3.12}$$

Equations (3.9), (3.11), and (3.12) can be formally solved with the aid of the fundamental matrix $\Phi(t)$, that is the matrix of solutions of the homogeneous equation $\dot{\Phi}(t) = C_t \Phi(t)$, $\Phi(0) = I$. For example, if $C_t = C$, in steady state, $\Phi(t) = \exp(Ct)$. Using $\Phi(t)$ we find

$$V(t) = \Phi(t)\left(V(0) + \int_0^t (\Phi(s))^{-1} D_s dW_s\right)$$

$$\mu_t = \Phi(t) E(V(0)) \tag{3.13}$$

$$K(s,t) = E((V(s)-\mu_s)(V(t)-\mu_t)')$$

$$= \Phi(s)\left(\Sigma_0 + \int_0^{\min(s,t)} (\Phi(u))^{-1} D_u D_u' ((\Phi(u))^{-1})' du\right)\Phi(t)'.$$

letting $s = t$, $k(s,t) = \Sigma_t$ thus providing a formula for Σ_t.

We note that in practice it will often be convenient to apply a computer routine for solving first-order differential equations directly to (3.11) and (3.12).

Suppose now we assume $C_t = C$ and $D_t = D$, that is we are in steady state. It will be shown in the appendix that all $k+1$ eigenvalues of C have negative real parts. In fact, the matrix C is nearly identical to the matrix A examined earlier. Then if $V(0) \sim N(0, \Sigma)$ with $\Sigma = \int_0^\infty e^{Ct} DD' e^{C't} dt$, $V(t)$ is a stationary Gaussian process with $E(V(t)) = 0$, $E(V(t)(V(t))') = \Sigma$

where $\underset{\sim}{\Sigma}$ is as defined above or is, equally, the unique nonnegative definite solution of the equation

$$\underset{\sim}{C}\underset{\sim}{\Sigma} + \underset{\sim}{\Sigma}\underset{\sim}{C}' = -\underset{\sim}{D}\underset{\sim}{D}'.$$

Furthermore, the covariance function $K(s,t) = H(s-t)$ is given by

$$H(s-t) = \begin{cases} e^{\underset{\sim}{C}(s-t)}\,\underset{\sim}{\Sigma} & s \geq t \geq 0 \\ \underset{\sim}{\Sigma}\, e^{\underset{\sim}{C}'(t-s)} & t \geq s \geq 0\ . \end{cases} \qquad (3.14)$$

We summarize our description of the $k+1$ dimensional queueing system as follows using the diffusion approximation:

$$(Q_1(t),\ldots,Q_k(t),R(t)) \cong$$

$$C(q_1(t),\ldots,q_k(t),r(t)) + \sqrt{C}(X_1(t),\ldots,X_k(t),Y(t))$$

where the first term is given by (3.4) and the second is a multivariate Ornstein-Uhlenbeck process with mean $\underset{\sim}{\mu}_t$, covariance function $H(s-t)$ as described earlier.

<u>Results for the Single Service Compartment</u>

We give the exact formulas in the steady state case for the special case of $k = 1$, a single service compartment. In this case, the deterministic approximations are still given by (3.6). The noise approximation will be a bivariate Ornstein-Uhlenbeck process with mean $\underset{\sim}{0}$. The covariance matrix $\underset{\sim}{\Sigma}$ will be the unique symmetric nonnegative definite solution of the equation

$$A\Sigma + \Sigma A' = -BB'. \tag{3.15}$$

where

$$A = \begin{pmatrix} -(\mu+\lambda+\alpha\nu r) & (1-q)\alpha\nu \\ \lambda + \alpha\nu r & -(1-\alpha q)\nu \end{pmatrix},$$

$$B = \begin{pmatrix} \sqrt{\lambda(1-q)} & \sqrt{\alpha\nu r(1-q)} & -\sqrt{\mu q} \\ \sqrt{\lambda q} & -\sqrt{(1-\alpha q)\nu r} & 0 \end{pmatrix}; \quad BB' = \begin{pmatrix} b_1 & b_3 \\ b_3 & b_2 \end{pmatrix}$$

for notational simplicity in what follows. Lastly

$$\Sigma = (\sigma_{ij}) \qquad \sigma_{11} = \mathrm{Var}(X(t))$$

$$\sigma_{22} = \mathrm{Var}(Y(t))$$

$$\sigma_{12} = \mathrm{Cov}(X(t),Y(t))$$

Solving (3.15) we find

$$\sigma_{11} = [b_1(|A|+a_{22}^2) - 2a_{12}a_{22}b_2 + a_{12}^2 b_3]/D$$

$$\sigma_{12} = [-a_{21}a_{22}b_1 + 2b_2a_{11}a_{22} - b_3a_{11}a_{12}]/D$$

$$\sigma_{22} = [a_{21}^2 b_1 - 2a_{11}a_{21}b_2 + b_3(|A|+a_{11}^2]/D$$

$$D = 2(a_{11}+a_{22})|A|. \tag{3.16}$$

In the last section we present a comparison of the theoretical calculations and simulation results in a variety of situations with $c = 10$ and 20. We are approximating the state of the system $(Q(t),R(t))$ by $c(q,r) + \sqrt{c}(X(t),Y(t))$. The diffusion approximation allows for more than a description of the marginal behavior in that the transient behavior can be characterized and the joint behavior at any set of time point, $(t_1,...,t_n)$ can be worked out using the multivariate Ornstein-Uhlenbeck process. However, we do not present numerical results of this kind in this paper.

4. The Spectral Matrix.

We now assume the system is in steady state and adopt the steady state diffusion approximation of it. The noise process is given by

$$d\underset{\sim}{U}(t) = \underset{\sim}{A}\underset{\sim}{U}(t)dt + \underset{\sim}{B}d\underset{\sim}{W}(t). \tag{4.1}$$

The noise process $\{U(t), t \geq 0\}$ is stationary, hence $(Q_1(t),\ldots,R(t))$ will also be stationary. This observation opens up the possibility of applying techniques from the theory of stationary processes to our queueing process. In particular we compute the spectral matrix associated with the noise process.

We begin with the spectral representations of the stationary processes $\underset{\sim}{U}(t)$ and $\underset{\sim}{W}(t)$

$$\underset{\sim}{U}(t) = \int e^{i\omega t}\, d\underset{\sim}{S}_{\underset{\sim}{U}}(\omega), \quad \underset{\sim}{W}(t) = \int e^{i\omega t}\, d\underset{\sim}{S}_{\underset{\sim}{W}}(\omega) \tag{4.2}$$

where $\{\underset{\sim}{S}_{\underset{\sim}{U}}(\omega), \omega \in (-\infty,\infty)\}$ and $\{\underset{\sim}{S}_{\underset{\sim}{W}}(\omega), \omega \in (-\infty,\infty)\}$ are processes with orthogonal increments. The spectral matrices associated with U and $\underset{\sim}{W}$, $\underset{\sim}{f}_{\underset{\sim}{U}}$ and $\underset{\sim}{f}_{\underset{\sim}{W}}$, are the matrices of cross spectral densities of the stationary processes and are given by

$$\underset{\sim}{f}_{\underset{\sim}{U}}^{(\omega)} = E(d\underset{\sim}{S}_{\underset{\sim}{U}}(\omega)\overline{d\underset{\sim}{S}_{\underset{\sim}{U}}}(\omega)), \quad \underset{\sim}{f}_{\underset{\sim}{W}}(\omega) = E(d\underset{\sim}{S}_{\underset{\sim}{W}}(\omega)\overline{d\underset{\sim}{S}_{\underset{\sim}{W}}}(\omega)) = \frac{\underset{\sim}{I}}{2\pi}.$$

We wish to compute $\underset{\sim}{f}_{\underset{\sim}{U}}(\omega)$ in terms of $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

Combining (4.1) and (4.2) by differentiation of (4.2) we find

$$(i\omega\underset{\sim}{I}-\underset{\sim}{A})\, d\underset{\sim}{S}_{\underset{\sim}{U}} = \underset{\sim}{B}\, d\underset{\sim}{S}_{\underset{\sim}{W}} \tag{4.3}$$

Taking transposes in (4.4), multiplying, and taking expectations we obtain

$$E[(i\omega \underset{\sim}{I}-\underset{\sim}{A})\, d\underset{\sim}{S}_{\underset{\sim}{U}}\, \overline{d\underset{\sim}{S}_{\underset{\sim}{U}}}\, (\overline{i\omega \underset{\sim}{I}-\underset{\sim}{A}})] = E[\underset{\sim}{B}\, d\underset{\sim}{S}_{\underset{\sim}{W}}\, \overline{d\underset{\sim}{S}_{\underset{\sim}{W}}}\, \underset{\sim}{B}'] = \frac{\underset{\sim}{BB}'}{2\pi} . \qquad (4.4)$$

Solving (4.5) for $\underset{\sim}{f}_{\underset{\sim}{U}} = E(d\underset{\sim}{S}_{\underset{\sim}{U}}\overline{d\underset{\sim}{S}_{\underset{\sim}{U}}})$ we find

$$\underset{\sim}{f}_{\underset{\sim}{U}}(\omega) = \frac{1}{2\pi}(\underset{\sim}{A}-i\omega\underset{\sim}{I})^{-1}\underset{\sim}{BB}'(\underset{\sim}{A}'+i\omega\underset{\sim}{I})^{-1}. \qquad (4.5)$$

We illustrate this computation in the case $k = 1$, a single service compartment. Here

$$\underset{\sim}{A} = \begin{pmatrix} -(\mu+\lambda+\alpha\nu r) & (1-q)\alpha\nu \\ \lambda+\alpha\nu r & -(1-\alpha q)\nu \end{pmatrix}, \quad \underset{\sim}{BB}' = \begin{pmatrix} 2\mu q & \Delta \\ \Delta & 2\lambda q \end{pmatrix}$$

with $\Delta = \lambda\sqrt{q(1-q)} - \nu r\sqrt{\alpha(1-q)(1-\alpha q)}$.

After a few simple matrix inversions and multiplications we compute

$$\underset{\sim}{f}_{\underset{\sim}{U}}(\omega) = \begin{pmatrix} f_{QQ}(\omega) & f_{QR}(\omega) \\ f_{RQ}(\omega) & f_{RR}(\omega) \end{pmatrix}$$

where

$$f_{QQ}(\omega) = [2\mu q\omega^2 + \nu(1-\alpha q)(2\mu q\nu(1-\alpha q)+\Delta\alpha\nu(1-q)) + \alpha\nu(1-q)(\Delta\nu(1-\alpha q) + 2\lambda q\alpha\nu(1-q)]/D$$

$$f_{RR}(\omega) = [2\lambda q\omega^2 + (\lambda+\alpha\nu r)2\mu q(\lambda+\alpha\nu r+\Delta(\mu+\lambda+\alpha\nu r)) + (\mu+\lambda+\alpha\nu r)(\Delta(\lambda+\alpha\nu r) + 2\lambda q(\mu+\lambda+\alpha\nu r)]/D \qquad (4.6)$$

$$f_{QR}(\omega) = \{[\Delta\omega^2 + \nu(1-\alpha q)(2\mu q(\lambda+\alpha\nu r) + \Delta(\mu+\lambda+\alpha\nu r)) + \alpha\nu(1-q)(\Delta(\lambda+\alpha\nu r) + 2\lambda q(\mu+\lambda+\alpha\nu r))] + i\omega[\Delta\nu(1-\alpha q) + 2\lambda q\alpha\nu(1-q) - 2\mu q(\lambda+\alpha\nu r) - \Delta(\mu+\lambda+\alpha\nu r)]\}/D$$

$$f_{RQ}(\omega) = \overline{f_{QR}(\omega)}$$

with

$$D = [\omega^2 + \alpha\nu(1-q)(\lambda+\alpha\nu r) - \lambda(1-\alpha q)(\mu+\lambda+\alpha\nu r)]^2 + [\nu(1-\alpha q) + \mu + \lambda + \alpha\nu r]^2\omega^2 .$$

The functions $f_{QQ}(\omega)$, $f_{RR}(\omega)$ are the spectral densities of $Q(t)$ and $R(t)$ respectively. The real part of $f_{QR}(\omega)$ is the cospectral density and the imaginary part is the quadrature spectral density. The latter two densities provide information about the phase behavior of $(Q(t),R(t))$. Notice that $f_{QQ}(\omega)$ and $f_{RR}(\omega)$ exhibit tail behavior ω^{-2} and thus fluctuate in a high-frequency manner similar to that of the ordinary Ornstein-Uhlenbeck process.

5. Comparison with Simulation

The previously quoted results are derived under the supposition that c, the number of service channels, becomes indefinitely large. Since it will be desirable to apply the approximations in case c is finite, we have undertaken to perform several simulations of particular systems with moderate c-values, and thus to provide an empirical check of model accuracy. We shall see that our approximations are generally good.

The simulation simulated is the system $(Q(t),R(t),L(t))$ in which $k = 1$ (the single service compartment case), and λ, μ, and ν are constants. By virtue of the Markov nature of the system we see that (i) sojourns in states are of independent and exponentially distributed duration, and (ii) state changes occur in accordance with multinomial Bernoulli trials. Reference to (3.1) shows that the state-dependent transition rates are the following

$$\begin{aligned}(Q(t),R(t),L(t)) &\to (Q(t)+1,R(t),L(t)) &&: \lambda c[1-Q(t)/c]\\ &\to (Q(t)+1,R(t)-1,L(t)) &&: \nu\alpha R(t)[1-Q(t)/c]\\ &\to (Q(t)-1,R(t),L(t)+1) &&: \mu Q(t)\\ &\to (Q(t),R(t)+1,L(t)) &&: \lambda c Q(t)/c\\ &\to (Q(t),R(t)-1,L(t)+1) &&: \nu(1-\alpha)R(t)\end{aligned} \tag{5.1}$$

Hence, given that at time t the system is in state $(Q(t),R(t),L(t))$ it resides there for independently and exponentially distributed times with means equal to the inverse of the sum of the right-hand side of (5.1), and then instantaneously jumps to a new, neighboring, state with probabilities given by the right side of (5.1) times the mean sojourn time in state. Our simulation program is based on this scheme.

The simulation results recorded were the fractions of times that the system inhabited each state $(Q=i, R=j, L=k)$ over the course of the simulation, the latter chosen to be long. These fractions approximate the stationary state probabilities. The latter probability estimates were in turn used to compute estimates of $E[Q(\infty)]$, $Var[Q(\infty)]$, $E[R(\infty)]$, $Var[R(\infty)]$, and $E[L(\infty)]$. Also, the tabulated empirical marginal distributions of simulated Q and R were plotted on normal probability paper in order to provide a visual check for normality.

Discussion

Agreement of the diffusion model with the simulation is, apparently, quite acceptable for the cases considered, which by design include relatively small numbers of channels. Extensive additional simulation results, left unreported here, convey the same message. The most noticeable discrepancy occurs in the estimates of $Var[R(\infty)]$: that obtained from the analytical approximation consistently exceeds the simulation estimate. Attempts to show that simulation's failure to reach steady state (starts were normally made at $Q(0) = R(0) = 0$) by starting higher gave essentially the same results. The discrepancy remains unexplained.

In order to aid quick appraisals we have tabulated some key quantiles; on the simulation side these are inexact both because of the inherent discreteness of the distributions and because of simulation sampling error, and on the diffusion side because of the use of the continuous normal approximation to a discrete distribution. We have, for example, diffusion approximated $Q_{0.95}$ by $E[Q] + 1.65$ St.Dev.[Q], $Q_{0.75}$ by $E[Q_{0.75}] + 0.68$ Std.Dev.$[Q_{0.75}]$, and $Q_{0.50}$ by $E[Q]$. The selected probabilities were calculated using a simple continuity correction, e.g.

$$P\{Q \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x+\frac{1}{2}-E[Q])/\text{Std.Dev.}[Q]} e^{-\frac{1}{2}z^2} dz$$

The latter give at a glance an impression of the Gaussian marginal model adequacy, which by and large is good out to the two-sigma level. As is to be anticipated, the Gaussian approximation degenerates in quality when $E[Q] + k$ Std.Dev.[Q] nears either zero or c.

Table 1

Comparison of Simulation and Diffusion Approximation

$\lambda = 7, \quad \mu = 4, \quad \nu = 6, \quad \alpha = 0.5, \quad c = 10$

	Simulation	Diffusion Approximation
EQQ(∞)]	7.31	7.34
Var[Q(∞)]	2.00	2.08
Std.Dev.[Q(∞)]	1.41	1.44
E[R(∞)]	13.55	13.54
Var[R(∞)]	16.84	20.80
Std.Dev.[R(∞)]	4.10	4.56
$Q_{0.05}$	4	5
$Q_{0.25}$	6	6
$Q_{0.50}$	7	7
$Q_{0.75}$	8	8
$Q_{0.95}$	9	10
$R_{0.05}$	7	6
$R_{0.25}$	10	10
$R_{0.50}$	13	14
$R_{0.75}$	16	17
$R_{0.95}$	20	21
$P\{Q(\infty) \le 9\}$	.956	.933
$P\{Q(\infty) \le 7\}$	.531	.544
$P\{Q(\infty) \le 5\}$	.104	.102
$P\{R(\infty) \le 20\}$	.946	.937
$P\{R(\infty) \le 13\}$	.518	.496
$P\{R(\infty) \le 6\}$	.030	.062

Table 2

Comparison of Simulation and Diffusion Approximation

$\lambda = 5, \quad \mu = 4, \quad \nu = 6, \quad \alpha = 0.5, \quad c = 10$

	Simulation	Diffusion Approximation
$E[Q(\infty)]$	6.45	6.49
$Var[Q(\infty)]$	2.36	2.51
$Std.Dev.[Q(\infty)]$	1.54	1.58
$E[R(\infty)]$	8.09	8.01
$Var[R(\infty)]$	10.20	12.65
$Std.Dev.[R(\infty)]$	3.19	3.56
$Q_{0.05}$	3	4
$Q_{0.25}$	5	5
$Q_{0.50}$	6	6
$Q_{0.75}$	7	8
$Q_{0.95}$	8	9
$R_{0.05}$	3	2
$R_{0.25}$	5	6
$R_{0.50}$	8	8
$R_{0.75}$	10	10
$R_{0.95}$	13	14
$P\{Q(\infty) \leq 9\}$	.986	.971
$P\{Q(\infty) \leq 6\}$	.497	.504
$P\{Q(\infty) \leq 3\}$	.030	.030
$P\{R(\infty) \leq 14\}$	.966	.965
$P\{R(\infty) \leq 8\}$	.581	.555
$P\{R(\infty) \leq 2\}$	.021	.050

Table 3

Comparison of Simulation and Diffusion Approximation

$\lambda = 7, \quad \mu = 4, \quad \nu = 6, \quad \alpha = 0.5, \quad c = 20$

	Simulation	Diffusion Approximation
$E[Q(\infty)]$	14.62	14.69
$Var[Q(\infty)]$	4.08	4.16
$Std.Dev.[Q(\infty)]$	2.02	2.04
$E[R(\infty)]$	27.01	27.08
$Var[R(\infty)]$	34.07	41.60
$Std.Dev.[R(\infty)]$	5.84	6.45
$Q_{0.05}$	11	11
$Q_{0.25}$	13	13
$Q_{0.50}$	14	15
$Q_{0.75}$	16	16
$Q_{0.95}$	17	18
$R_{0.05}$	17	16
$R_{0.25}$	22	23
$R_{0.50}$	26	27
$R_{0.75}$	30	31
$R_{0.95}$	36	38
$P\{Q(\infty) \leq 19\}$	.998	.991
$P\{Q(\infty) \leq 15\}$	.654	.655
$P\{Q(\infty) \leq 11\}$	.066	.059
$P\{R(\infty) \leq 40\}$	.982	.981
$P\{R(\infty) \leq 27\}$	.532	.526
$P\{R(\infty) \leq 14\}$	.026	.025

Table 4

Comparison of Simulation and Diffusion Approximation

$\lambda = 5, \quad \mu = 4, \quad \nu = 6, \quad \alpha = 0.5, \quad c = 20$

	Simulation	Diffusion Approximation
$E[Q(\infty)]$	12.89	12.98
$Var[Q(\infty)]$	4.89	5.02
$Std.Dev.[Q(\infty)]$	2.21	2.24
$E[R(\infty)]$	15.98	16.02
$Var[R(\infty)]$	20.58	25.30
$Std.Dev.[R(\infty)]$	4.54	5.03
$Q_{0.05}$	9	9
$Q_{0.25}$	11	11
$Q_{0.50}$	12	13
$Q_{0.75}$	14	15
$Q_{0.95}$	16	17
$R_{0.05}$	8	8
$R_{0.25}$	12	13
$R_{0.50}$	15	16
$R_{0.75}$	18	19
$R_{0.95}$	23	24
$P\{Q(\infty) \leq 17\}$	.989	.978
$P\{Q(\infty) \leq 13\}$	.596	.591
$P\{Q(\infty) \leq 9\}$	.066	.061
$P\{R(\infty) \leq 24\}$	.960	.954
$P\{R(\infty) \leq 16\}$	.569	.540
$P\{R(\infty) \leq 8\}$	.037	.067

Appendix

We prove that the matrices $\underset{\sim}{A}$ from (3.7) and $\underset{\sim}{C}_t$ from (3.9) have eigenvalues with strictly negative real parts. Both matrices have the form

$$\underset{\sim}{M} = \begin{pmatrix} -(\mu_1+c) & -c & -c & & -c & d \\ \mu_1 & -\mu_2 & 0 & & 0 & 0 \\ 0 & \mu_2 & -\mu_3 & \cdots & \vdots & \vdots \\ \vdots & 0 & \mu_3 & & \vdots & \vdots \\ \vdots & \vdots & 0 & & 0 & \vdots \\ 0 & 0 & 0 & & -\mu_k & 0 \\ c & c & c & & c & e \end{pmatrix}$$

with $c = \lambda$, $d = \alpha\nu$, $e = \nu$ for $\underset{\sim}{A}$ and $c = \lambda + \alpha\nu r$, $d = \alpha\nu(1-q)$, $e = \nu(1-\alpha q)$ for $\underset{\sim}{C}_t$. In both cases $e - d = \nu(1-\alpha) > 0$.

We wish to solve for the $k+1$ roots of the equation $|\underset{\sim}{M} - \theta\underset{\sim}{I}| = 0$. Simple manipulation gives

$$|M - \theta I| = \begin{vmatrix} -(\theta+\mu_1) & 0 & & 0 & -(\theta+e-d) \\ \mu_1 & -(\theta+\mu_2) & \cdots & \vdots & 0 \\ 0 & \mu_2 & & \vdots & \vdots \\ \vdots & 0 & & 0 & \vdots \\ 0 & 0 & & -(\theta+\mu_k) & 0 \\ c & c & & c & -(\theta+e) \end{vmatrix}$$

$$= (-1)^{k+1} (\theta+e) \prod_{i=1}^{k} (\theta+\mu_i) + (-1)^{k+3} (\theta+e-d) D_k$$

where

$$D_k = c \begin{vmatrix} \mu_1 & -(\theta+\mu_2) & \cdots & 0 & 0 \\ 0 & \mu_2 & & \vdots & \vdots \\ \vdots & 0 & & 0 & \vdots \\ 0 & 0 & & -(\theta+\mu_{k-1}) & 0 \\ & & & \mu_{k-1} & -(\theta+\mu_k) \\ 1 & 1 & & 1 & 1 \end{vmatrix}$$

The determinant D_k can be computed recursively as

$$D_k = (\theta+\mu_k)D_{k-1} + \prod_{i=1}^{k-1} \mu_i,$$

and these equations can be solved. We find

$$D_k = \left[\prod_{i=1}^{k} (\theta+\mu_i) - \prod_{i=1}^{k} \mu_i\right]/\theta. \tag{A.1}$$

Substitution of (A.1) into $|\underset{\sim}{M} - \theta\underset{\sim}{I}|$ gives

$$0 = (\theta+e) \prod_{i=1}^{k} (\theta+\mu_i) + c(\theta+e-d)\left(\prod_{i=1}^{k} (\theta+\mu_i) - \prod_{i=1}^{k} \mu_i\right)/\theta. \tag{A.2}$$

We wish to show that all $k+1$ roots of (2) have strictly negative real parts assuming e, $e-d$, $\mu_1, \ldots, \mu_k$ are all strictly positive.

The $k+1$ degree polynomial on the right side of equation (A.2) has strictly positive coefficients, thus by Descartes rule only strictly negative real roots are possible. It remains to consider the case of complex roots and we assume $\theta = a + b_i$. Since complex roots of (A.2) appear in conjugate pairs we assume $b > 0$ without loss in generality.

Assuming θ is complex allows us to rewrite (A.2) as

$$\theta(\theta+e) = c(\theta+e-d)\left(\prod_{i=1}^{k}\left(\frac{\mu_i}{\theta+\mu_i}\right) - 1\right). \tag{A.3}$$

We consider the case $a \geq 0$ or $0 < \arg\theta \leq \frac{\pi}{2}$. In this case

$$\left[\prod_{i=1}^{k}\left(\frac{\mu_i}{\theta+\mu_i}\right)\right] < 1, \quad \text{and it follows that}$$

$$\mathrm{Re}\left(\prod_{i=1}^{k}\left(\frac{\mu_i}{\theta+\mu_i}\right) - 1\right) < 0 \quad \text{or} \quad \frac{\pi}{2} < \arg\left(\prod_{i=1}^{k}\left(\frac{\mu_i}{\theta+\mu_i}\right) - 1\right) < \frac{3\pi}{2} .$$

Furthermore, $0 < \arg(\theta+e-d) < \arg(\theta+e) < \arg(\theta) \leq \frac{\pi}{2}$, and it follows that

$$0 < \arg(\theta(\theta+e)) < \arg\left[c(\theta+e-d)\left(\prod_{i=1}^{k}\left(\frac{\mu_i}{\theta+\mu_i}\right) - 1\right)\right] < 2\pi.$$

Consequently, θ cannot be a root of (A.3) and consequently, (A.2) if $a \geq 0$. We have just proved that any complex root must have strictly negative real part, thus all $k+1$ eigenvalues of M have strictly negative real parts.

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D. L. Iglehart 1
Department of Operations Research
Stanford University
Stanford, California 94305

J. Riordan 1
Department of Mathematics
Rockefeller University
New York, New York 10021

G. Fishman 1
University of North Carolina
Chapel Hill, North Carolina 27514

P. T. Holmes 1
Department of Mathematics
Clemson University
Clemson, South Carolina 29631

R. Elashoff 1
Biomathematics
University of California
San Francisco, California 94122

D. P. Gaver 25
Code 55
Naval Postgraduate School
Monterey, California 93940